How Good Is a Strategy in a Game with Nature?

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***Abstract*—We consider games with two antagonistic players — Éloïse (modelling a program) and Abélard (modelling a byzantine environment) — and a third, unpredictable and uncontrollable player, that we call Nature. Motivated by the fact that the usual probabilistic semantics very quickly leads to undecidability when considering either infinite game graphs or imperfect information, we propose two alternative semantics that leads to decidability where the probabilistic one fails: one based on counting and one based on topology.**

I. Introduction

An important problem in computer science is the specification and the verification of systems allowing **non-deterministic** behaviours. A non-deterministic behaviour can appear in several distinct contexts: (i) controllable behaviours (typically arising when the **program** is not fully specified, permitting to later restrict it); (ii) uncontrollable possibly byzantine behaviours (typically arising from interactions of the program with its **environment**, *e.g.* a user); (iii) uncontrollable unpredictable behaviours (usually arising from **nature** often modelled by randomisation). Here we do an explicit distinction between environment and nature: while we cannot assume that a user will not be malicious, the situation with nature is different as we can accept a negligible set of bad behaviours as it implicitly means that they are very unlikely to appear. On top of this, one may also want to allow **imperfect information** (typically arising when the protagonists — program, nature and environment — share some public variables but also have their own private variables) and/or **infinite** state systems (typically arising when modelling recursive procedures).

Two-player stochastic games on graphs are a natural way to model such a system. In a nutshell, a stochastic game is defined thanks to a directed graph whose vertices have been partitioned among two antagonistic players — Éloïse (modelling the program) and Abélard (modelling the byzantine environment) — and a third, unpredictable and uncontrollable player, that we call Nature. The play starts with a token on a fixed initial vertex of the graph that is later moved by the players (the player owning the vertex where the token is, chooses a neighbour to which the token is moved to, and so one forever) leading to an infinite path in the game graph. We are interested in ***zero-sum*** games, *i.e.* we consider a winning condition Ω consisting of a subset of plays and we say that a play is winning for Éloïse if it belongs to Ω and otherwise it is winning for Abélard. A game $\mathbb{G}$ is such a graph together with a winning condition.

In the previous model, Nature usually comes with a probabilistic semantics (as in the seminal work of Condon [1]), *i.e.* any vertex controlled by Nature is associated with a probability distribution over its neighbours and this probability distribution is used to pick the next move when the token is on the corresponding vertex. The central concept is the one of a strategy, which maps to any prefix of a play the next vertex to move the token to. Once a strategy φ_{E} for Éloïse and a strategy φ_{A} for Abélard have been fixed, the set of all possible plays in the game where the players respect their strategies can be equipped with a probability measure $\mu_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$, and one can therefore define the ***value*** of the game as (φ_{E} and φ_{A} range over Éloïse and Abélard strategies respectively)

$$\mathrm{Val}(\mathbb{G}) = \sup_{\varphi_{\mathrm{E}}} \inf_{\varphi_{\mathrm{A}}} \{\mu_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}(\Omega)\}$$

Then, the following questions are of special interest.

1) "*Decide whether the value of the game is larger than some given threshold* η" and its qualitative weakening "*Decide if the value is equal to* 1".
2) "*When exists, compute an optimal strategy*[1] *for Éloïse*".

If the game is played on a finite graph and the winning condition is ω-regular, all those questions can be answered and algorithms are known and their complexities, depending on the winning condition, range from P to PSPACE (see *e.g.* [2] for an overview).

Unfortunately the landscape drastically changes as soon as one either considers infinite game graphs and/or imperfect information (*i.e.* instead of observing the exact state of the system, each player only observes that it belongs to some equivalence class). In particular we have the following undecidability (somehow minimal) results:

- If the game graph is a pushdown graph, then even if Abélard is not part of the game, the qualitative analysis of reachability games is undecidable [3].
- If Éloïse has imperfect information then, even if the graph is finite and Abélard is not part of the game, the qualitative analysis of co-Büchi games is undecidable [4].

In this paper, we propose two alternative semantics that lead to decidable problems where the previous probabilistic approach fails. The main idea is to evaluate (for fixed strategies of Éloïse and Abélard) how "small" is the set of resulting loosing plays for Éloïse.

Our first setting is based on **counting**. In order to evaluate how good a situation is for Éloïse (*i.e.* using some strategy φ_{E} against a strategy φ_{A} of Abélard) we simply count how

[1] An optimal strategy φ_{E} for Éloïse is one such that $\mathrm{Val}(\mathbb{G}) = \inf_{\varphi_{\mathrm{A}}}\{\mu_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}(\Omega)\}$. Note that it may not exist even if the graph is finite.

many loosing plays there are: the fewer the better. Of special interest are those strategies for which, against any strategy of Abélard, the number of loosing plays is at most countable. The idea of counting can be traced back to the work in [5] on automata with cardinality constraints. There is also work on the logical side with decidable results but that do not lead to efficient algorithms [6].

Our second setting is based on **topology**. In order to evaluate how good a situation is for Éloïse (*i.e.* using some strategy φ_{E} against a strategy φ_{A} of Abélard) we use a topological notion of "bigness"/"smallness" given by the concept of large/meager set. The idea of using topology was considered previously in the context of *finite* Markov chains [7] and *finite* Markov decision processes [8].

We investigate both the perfect (Section III) and the imperfect information setting (Section IV) and consider game graphs with countably many vertices. For each setting (imposing to have only Éloïse and Nature for topological setting, and parity condition when handling imperfect information) we give a reduction to a game *without Nature* that characterises those games where Éloïse has a "good" strategy.

Finally, in Section V we do a comparison with previous works and give several consequences. Among others, we derive the main results of [8], the ones on some of the variants of tree automata considered in [5], [9], as well as decidability for Éloïse-Abélard-Nature pushdown games with an unboundedness condition on the stack.

II. Preliminaries

Let X be a set, we denote by $Card(X)$ its cardinal. In this work all cardinals will be either finite, equal to $\aleph_0$ (the cardinality of the natural numbers) or equal to $2^{\aleph_0}$ (the cardinality of the real numbers). A set is ***countable*** if its cardinal is smaller or equal than $\aleph_0$ (equivalently, the set is either finite or in bijection with the natural numbers).

Let A be a (possibly infinite) set seen here as an *alphabet*. We denote by A^* the set of finite words over the alphabet A and by A^ω the set of infinite words over the alphabet A. If u is a word we denote by $|u| \in \mathbb{N} \cup \{\omega\}$ its length. We denote by ε the empty word and let $A^+ = A^* \backslash \{\varepsilon\}$. If $u \in A^*$ and $v \in A^* \cup A^\omega$ we denote by $u \cdot v$ (or simply uv) the (possibly infinite) word obtained by concatenating u and v. A word $u \in A^*$ is a *prefix* of a word $w \in A^* \cup A^\omega$ if there exists some $v \in A^* \cup A^\omega$ such that $w = u \cdot v$, and we denote this situation by $u \sqsubseteq w$; moreover if $u \neq w$ we say that u is a *strict* prefix (denoted by $u \sqsubset w$). A set $S \subseteq A^*$ is ***prefix-closed*** if for all $u \in S$ and $v \sqsubseteq u$ one has $v \in S$. Now let $(u_i)_{i \geqslant 0}$ be a sequence of finite words in A^* such that for all $i \geqslant 0$ one has $u_i \sqsubseteq u_{i+1}$ and for infinitely many $i \geqslant 0$ one has $u_i \sqsubset u_{i+1}$. Then we can define its ***limit*** $u_\infty \in A^\omega$ as the unique infinite word such that for all $i \geqslant 0$, $u_i \sqsubset u_\infty$.

In this paper we consider various notions of trees that we introduce now. Let D be a (countable) set of directions; a D-tree (or simply a ***tree*** when D is clear) is a prefix-closed subset of D^*. A D-tree is ***complete*** if it equals D^*; it is ***binary*** if $Card(D) = 2$ (and in general one identifies D with $\{0, 1\}$). If T is a tree, we refer to $u \in T$ as a ***node***; if $T = \{0, 1\}^*$ is the complete binary tree we refer to $u \cdot 0$ (resp. $u \cdot 1$) as the left (resp. right) son of u. The node ε is called the root.

An (infinite) ***branch*** in a D-tree T is an infinite word $\pi \in D^\omega$ such that there is an increasing (for the prefix ordering) sequence of nodes $(u_i)_{i \geqslant 0}$ whose limit is π. A node u belongs to a branch π whenever $u \sqsubset \pi$. Branches in the complete D-tree are exactly D^ω. For a node $u \in T$, the cone $\mathrm{Cone}_T(u)$ is defined as the set of branches of T passing through u.

Let A be a (countable) alphabet; an ***A-labelled tree*** t is a total function $t : Dom \to A$ where Dom is a tree. For a node $u \in Dom$ we call $t(u)$ the label of u; and for a branch $\pi = \pi_0\pi_1 \cdots$ of the tree Dom we call $t(\pi_0)t(\pi_0\pi_1)t(\pi_0\pi_1\pi_2) \cdots \in A^\omega$ the label of π. For a node $u \in Dom$ we let $t[u]$ be the subtree rooted at u, *i.e.* $t[u] : \mathrm{Dom}' \to A$ with $\mathrm{Dom}' = \{v \mid u \cdot v \in \mathrm{Dom}\}$ and $t[u](v) = t(uv)$.

III. Perfect Information Games With Nature

A (directed) ***graph*** G is a pair (V, E) where V is a countable set of ***vertices*** and $E \subseteq V \times V$ is a set of ***edges***. For a vertex v we denote by $E(v)$ the set of its successors $\{v' \mid (v, v') \in E\}$ and in the rest of the paper (hence, this is implicit from now on), we only consider graphs that have no dead-end, *i.e.* such that $E(v) \neq \varnothing$ for all v.

We are interested in this work by games involving two antagonistic players — Éloïse and Abélard — together with a third *uncontrollable and unpredictable* player called Nature. An ***arena*** is a tuple $\mathcal{G} = (G, V_{\mathrm{E}}, V_{\mathrm{A}}, V_{\mathrm{N}})$ where $G = (V, E)$ is a graph and $V = V_{\mathrm{E}} \uplus V_{\mathrm{A}} \uplus V_{\mathrm{N}}$ is a partition of the vertices among the protagonists. We say that a vertex v is owned by Éloïse (resp. by Abélard, resp. by Nature) if $v \in V_{\mathrm{E}}$ (resp. $v \in V_{\mathrm{A}}$, resp. $v \in V_{\mathrm{N}}$).

Éloïse, Abélard and Nature play in $\mathcal{G}$ by moving a pebble along edges. A ***play*** from an initial vertex v_0 proceeds as follows: the player owning v_0 moves the pebble to a vertex $v_1 \in E(v_0)$. Then the player owning v_1 chooses a successor $v_2 \in E(v_1)$ and so on. As we assumed that there is no dead-end, a play is an infinite word $v_0v_1v_2 \cdots \in V^\omega$ such that for all $0 \leqslant i$, one has $v_{i+1} \in E(v_i)$. A ***partial play*** is a prefix of a play, *i.e.* it is a finite word $v_0v_1 \cdots v_\ell \in V^*$ such that for all $0 \leqslant i < \ell$, one has $v_{i+1} \in E(v_i)$.

A ***strategy*** for Éloïse is a function $\varphi_{\mathrm{E}} : V^*V_{\mathrm{E}} \to V$ assigning, to every partial play ending in some vertex $v \in V_{\mathrm{E}}$, a vertex $v' \in E(v)$. Strategies for Abélard are defined likewise, and usually denoted φ_{A}. In a given play $\lambda = v_0v_1 \cdots$ we say that Éloïse (resp. Abélard) ***respects a strategy*** φ if whenever $v_i \in V_{\mathrm{E}}$ (resp. $v_i \in V_{\mathrm{A}}$) one has $v_{i+1} = \varphi(v_0 \cdots v_i)$. A strategy φ is ***positional*** if for any two partial plays π and π' ending in the same vertex, we have $\varphi(\pi) = \varphi(\pi')$.

With an initial vertex v_0 and a pair of strategies $(\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}})$, we associate the set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}}$ of possible plays where each player respects his strategy, *i.e.* $\lambda \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}}$ if and only if λ is a play starting from v_0 where Éloïse respects φ_{E} and Abélard respects φ_{A}. In the classical setting where Nature is not present (*i.e.*, $V_{\mathrm{N}} = \varnothing$), when the strategies of Éloïse and Abélard are fixed there is only one possible play.

The presence of Nature induces a branching structure. In fact $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ is the set of branches of the V-tree $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ consisting of those partial plays where each player respects his strategy.

A ***winning condition*** is a subset $\Omega \subseteq V^\omega$ and a ***game*** is a tuple $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ consisting of an arena, a winning condition and an initial vertex v_0. In this paper, we only consider winning conditions that are ***Borel*** sets, *i.e.* that belong to the σ-algebra defined from the basics open sets of the form KV^ω with $K \subseteq V^*$.

A well known popular example of such a winning conditions are the ***parity conditions***. Let $\mathrm{Col} : V \to C$ be a colouring function assigning to any vertex a colour in a *finite* set $C \subset \mathbb{N}$. Then one defines Ω_{Col} to be the set of all plays where the smallest infinitely often repeated colour is even, *i.e.*

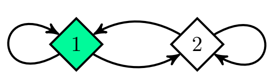
$$\Omega_{\mathrm{Col}} = \{v_0v_1v_2\cdots \in V^\omega \mid \liminf(\mathrm{Col}(v_i))_i \text{ is even.}\}$$

Büchi (resp. ***co-Büchi***) conditions are those parity conditions where $C = \{0, 1\}$ (resp. $C = \{1, 2\}$); it requires for a play to be winning to go infinitely (resp. only finitely) often through vertices coloured by 0 (resp. 1) and in general it is defined by a set of final (resp. forbidden) vertices: those of colour 0 (resp. 1).

A play λ from v_0 is ***won*** by Éloïse if and only if $\lambda \in \Omega$; otherwise λ is won by Abélard.

A strategy φ_{E} is ***winning*** for Éloïse in $\mathbb{G}$ if for any strategy φ_{A} of Abélard one has $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \subseteq \Omega$, *i.e.* she wins regardless of the choices of Abélard and Nature. Symmetrically, a strategy φ_{A} is ***winning*** for Abélard in $\mathbb{G}$ if for any strategy φ_{E} of Éloïse one has $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \cap \Omega = \varnothing$.

As the winning condition is Borel, it is a well known result — Martin's determinacy Theorem [10] — that whenever $V_{\mathrm{N}} = \varnothing$ the game is determined, *i.e.* either Éloïse or Abélard has a winning strategy. Due to Nature, it is easily seen that in many situations neither Éloïse nor Abélard has a winning strategy (from a given initial vertex). For instance consider the Büchi game above where all vertices belong to Nature and where the final vertex is coloured. The strategies for Éloïse and Abélard are both trivial and there are both winning (e.g. 1^ω) and loosing plays (e.g. 12^ω) for Éloïse.



One way of solving this situation, *i.e.* to still evaluate how good a strategy/game is for Éloïse, is to equip Nature with a probabilistic semantics, leading to the concept of *stochastic games* that we briefly recall in the next section, the main focus of the present paper being to propose alternative semantics (the cardinality one and the topological one) that lead to decidable problems where the previous probabilistic approach fails.

A. The Probabilistic Setting

We now briefly recall the concept of *stochastic games* [1], [11] (see also [2] for a recent overview of the field and formal details on the objects below) which consists of equipping the games with Nature with a probabilistic semantics. In a nutshell, any vertex in V_{N} comes with a probability distribution over its neighbours and then, for a fixed tuple $(v_0, \varphi_{\mathrm{E}}, \varphi_{\mathrm{A}})$, these probabilities are used to defined a σ-algebra (taking as cones the sets of plays sharing a common finite prefix) and a probability measure $\mu_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ on $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$. In particular, this permits to associate with any pair $(\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}})$ a real in $[0, 1]$ defined as the probability of $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \cap \Omega$ in the previous space. Of special interest is the ***value*** of a given strategy φ_{E} of Éloïse, that estimates how good φ_{E} is for her:

$$\mathrm{Val}_{\mathbb{G}}(\varphi_{\mathrm{E}}) = \inf\{\mu_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}(\Omega) \mid \varphi_{\mathrm{A}} \text{ Abélard strategy}\}$$

Finally, the value of the game is defined by taking the supremum of the values of Éloïse's strategies:

$$\mathrm{Val}(\mathbb{G}) = \sup\{\mathrm{Val}_{\mathbb{G}}(\varphi_{\mathrm{E}}) \mid \varphi_{\mathrm{E}} \text{ Éloïse strategy}\}$$

A strategy φ_{E} is optimal when $\mathrm{Val}_{\mathbb{G}}(\varphi_{\mathrm{E}}) = \mathrm{Val}(\mathbb{G})$ and it is almost surely winning when $\mathrm{Val}_{\mathbb{G}}(\varphi_{\mathrm{E}}) = 1$.

B. The Cardinality Setting

We now propose a change of perspective based on counting: in order to evaluate how good a situation is for Éloïse (*i.e.* using some strategy φ_{E} against a strategy φ_{A} of Abélard) we simply count how many loosing plays there are; the fewer they are the better the situation is.

First note the following proposition [12] that characterises the cardinals of the Borel subsets of some set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$.

Proposition 1. *For any arena, any initial vertex, any pair of strategies* $(\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}})$ *and any Borel subset* $S \subseteq \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$, *one has* $Card(S) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

We define the cardinality leaking of an Éloïse's strategy as a measure of its quality.

Definition 1 (Cardinality Leaking of a Strategy)**.** *Let* $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ *be a game and let* φ_{E} *be a strategy of Éloïse. The* ***cardinality leaking*** *of* φ_{E} *is the cardinal* $CardLeak(\varphi_{\mathrm{E}})$ *defined by*

$$CardLeak(\varphi_{\mathrm{E}}) = \sup\{Card(\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}\backslash\Omega) \mid \varphi_{\mathrm{A}} \text{ strategy of Abélard}\}$$

Proposition 1 implies that $CardLeak(\varphi_{\mathrm{E}}) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

The goal of Éloïse is to minimise the number of loosing plays, hence leading the following concept.

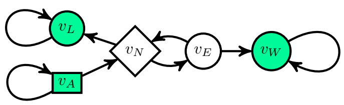
Definition 2 (Leaking Value of a Game)**.** *Let* $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ *be a game. The* ***leaking value*** *of* $\mathbb{G}$ *is defined by*

$$LeakVal(\mathbb{G}) = \inf\{CardLeak(\varphi_{\mathrm{E}}) \mid \varphi_{\mathrm{E}} \text{ strategy of Éloïse}\}$$

Thanks to Proposition 1 it follows that $LeakVal(\mathbb{G}) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Remark 1. *One can wonder whether the* sup *in the definition of* $CardLeak(\varphi_{\mathrm{E}})$ *can be replaced by a* max, i.e. *whether, against any fixed strategy of Éloïse, Abélard has always an "optimal" counter strategy.*

Actually this is not possible as exemplified by the Büchi game below — a circle (resp. square, resp. diamond) vertex *depicts one that belongs to Éloïse (resp. Abélard, resp. Nature) and coloured vertices are the final ones — with v_A as initial vertex.*



Consider the strategy φ_{E} of Éloïse consisting in a partial play ending in v_E (in v_L and v_W Éloïse has only one choice) to go to v_N if there are less occurrences of v_E than of v_A in the partial play and to go to v_W otherwise. Clearly for any strategy φ_{A} of Abélard, $Card(\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_j}\backslash\Omega)$ is finite. However for any $k \geqslant 0$, Abélard can ensure that there are k plays lost by Éloïse by looping $k-1$ times on the vertex v_A before going to v_N.

As cardinals are well-ordered, Éloïse always has an "optimal" strategy for the leaking value criterion (*i.e.* we can replace the $\inf$ by a $\min$ in Definition 2).

Proposition 2. *Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game. There is a strategy φ_{E} of Éloïse such that $LeakVal(\mathbb{G}) = CardLeak(\varphi_{\mathrm{E}})$.*

In the reminder of this article, we consider that a strategy is good from the cardinality point of view if its cardinality leaking is countable. From a modelisation point of view, we agree that this notion can be questionnable. In particular it only makes sense if for all strategy φ_{E} and φ_{A} of Éloïse and Abélard respectively, the set of outcomes is uncountable. A suffient condition to ensure this last property is that all vertices of Nature have at least two successors and that every play visits infinitely many vertices of Nature.

C. The Topological Setting

A notion of topological "bigness" and "smallness" is given by *large* and *meager* sets respectively (see [7], [13] for a survey of the notion). From the modelisation point of view, the intuition is that meager sets (the complements of large sets) are somehow negligible. In [7], the authors give weight to this idea by showing that, for regular trees (*i.e.* the unfolding of a finite graphs), the set of branches satisfying an ω-regular condition is large if and only if it has probability 1 (in the sense of Section III-A). However they also show that in general, even for the Büchi condition and when the tree is the unfolding of a pushdown graph, this is no longer true (see [7, p. 27]).

Let t be a D-tree for some set D of directions. Then its set of branches can be seen as a topological space by taking as basic open sets the set of cones. A set of branches $B \subseteq D^\omega$ is ***nowhere dense*** if for all node $u \in t$, there exists another node $v \in t$ such that $u \sqsubseteq v$ and such that v does not belong to any branch in B. A set of branches is ***meager*** if it is the countable union of nowhere dense sets. Finally it is ***large*** if it is the complement of a meager set.

A natural topological criterion to consider that a strategy φ_{E} for Éloïse is good against a strategy φ_{A} of Abélard is that the set of plays lost by Éloïse is meager in the tree $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$.

Definition 3 (Topologically Good Strategies)**.** *Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game and let φ_{E} be a strategy of Éloïse. We say that φ_{E} is **topologically good** if and only if for any strategy φ_{A} of Abélard the set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}\backslash\Omega$ of losing plays for Éloïse is meager in the tree $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$; or equivalently the set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \cap \Omega$ of plays won by Éloïse is large.*

Banach-Mazur theorem gives a game characterisation of large and meager sets of branches (see for instance [13], [14]). The ***Banach-Mazur game*** on t, is a two-player game where Abélard and Éloïse choose alternatively a node in the tree, forming a branch: Abélard chooses first a node and then Éloïse chooses a descendant of the previous node and Abélard chooses a descendant of the previous node and so on forever. In this game it is **always Abélard that starts** a play.

Formally a ***play*** is an infinite sequence $u_1, u_2, \ldots$ of words in D^+ such that for all i one has $u_1 u_2 \cdots u_i \in t$, and the branch associated with this play is $u_1 u_2 \cdots$. A ***strategy*** for Éloïse is a mapping $\varphi : (D^+)^+ \to D^+$ that takes as input a finite sequence of words, and outputs a word. A play $u_1, u_2, \ldots$ ***respects*** φ if for all $i \geqslant 1$, $u_{2i} = \varphi(u_1, \ldots, u_{2i-1})$. We define $Outcomes(\varphi)$ as the set of plays that respect φ and $\mathcal{B}(\varphi)$ as the set branches associated with the plays in $Outcomes(\varphi)$.

The Banach-Mazur theorem (see *e.g.* [13, Theorem 4]) states that a set of branches B is large if and only if there exists a strategy φ for Éloïse such that $\mathcal{B}(\varphi) \subseteq B$.

Furthermore a folk result (see *e.g.* [13, Theorem 9]) about Banach-Mazur games states that when B is Borel[2] one can look only at "simple" strategies, defined as follows. A ***decomposition-invariant strategy*** is a mapping $f : t \to D^+$ and we associate with f the strategy φ_f defined by $\varphi_f(u_1, \ldots, u_k) = f(u_1 \cdots u_k)$. Finally, we define $Outcomes(f) = Outcomes(\varphi_f)$ and $\mathcal{B}(f) = \mathcal{B}(\varphi_f)$. The folk result states that for any Borel set of branches B, there exists a strategy φ such that $Outcomes(\varphi) \subseteq B$ if and only if there exists a decomposition-invariant strategy f such that $\mathcal{B}(f) \subseteq B$.

D. A Game to Decide If the Leaking Value Is at Most $\aleph_0$

Our goal in this section is to design a technique to decide if the leaking value is at most $\aleph_0$ in a given two player game with Nature for an arbitrary Borel winning condition.

Fix an ***arena*** $\mathcal{G} = (G = (V, E), V_{\mathrm{E}}, V_{\mathrm{A}}, V_{\mathrm{N}})$ and a game $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ where Ω is a Borel winning condition. We design a two-player perfect information game without Nature $\widehat{\mathbb{G}} = (\widehat{\mathcal{G}}, v_0, \widehat{\Omega})$ such that Éloïse wins $\widehat{\mathbb{G}}$ if and only if $LeakVal(\mathbb{G}) \leqslant \aleph_0$.

Intuitively in the game $\widehat{\mathbb{G}}$, every vertex v of Nature is replaced by a gadget in which Éloïse announces a successor w of v (*i.e.* in $w \in E(v)$) that she wants to *avoid* and then Abélard chooses a successor of v. If he picks w we say that he *disobeys* Éloïse otherwise he *obeys* her. In vertices of Éloïse and Abélard, the game $\widehat{\mathbb{G}}$ works the same as the game $\mathbb{G}$.

[2] This statement holds as soon as the Banach-Mazur games are determined and hence, in particular for Borel sets.

The winning condition $\widehat{\Omega}$ for Éloïse is either that the play (without the gadget nodes) belongs to Ω or that Abélard does not obeys Éloïse infinitely often (*i.e.* after some point, Abélard always disobeys Éloïse). This is in particular the case if, after some point, no vertex corresponding to vertex of Nature is encountered.

Formally one defines $\widehat{G} = (\widehat{V} = \widehat{V}_{\mathrm{E}} \cup \widehat{V}_{\mathrm{A}}, \widehat{E})$ where $\widehat{V}_{\mathrm{E}} = V_{\mathrm{E}} \cup V_{\mathrm{N}}$, $\widehat{V}_{\mathrm{A}} = V_{\mathrm{A}} \cup \{(v, w) \mid v \in V_{\mathrm{N}} \text{ and } w \in E(v)\}$ and

$$\begin{aligned}\widehat{E} = E \setminus (V_{\mathrm{N}} \times V) \ &\cup\ \{(v, (v, w)) \mid v \in V_{\mathrm{N}} \text{ and } w \in E(v)\}\\ &\cup\ \{((v, w), v') \mid v \in V_{\mathrm{N}} \text{ and } v', w \in E(v)\}\end{aligned}$$

Intuitively, in a partial play ending in $v \in V_{\mathrm{N}}$, Éloïse chooses (v, w) for some $w \in E(v)$ to indicate that she wants to avoid w. Then Abélard in (v, w) chooses a successor of v knowing that if he picks w he is disobeying Éloïse.

For ease of presentation, we view a partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$ as a partial play π in $\mathbb{G}$ together with a mapping associating to every prefix of π ending in V_{N} (with the possible exception of π itself) the successor that Éloïse wishes to avoid.

Formally for a partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$, we denote by $[\![\hat{\pi}]\!]$ the partial play of $\mathbb{G}$ obtained by removing all occurrences of vertices in $V_{\mathrm{N}} \times V$ from $\hat{\pi}$. A partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$ is entirely characterised by the pair (π, ξ) where π is the partial play $[\![\hat{\pi}]\!]$ and ξ is the mapping such for all $\pi' \sqsubseteq \pi$, $\xi(\pi') = w$ if and only if there exists $\hat{\pi}' \sqsubseteq \hat{\pi}$ with $[\![\hat{\pi}']\!] = \pi'$ and $\hat{\pi}'$ ends in a vertex of the form (v, w) for some $v \in V_{\mathrm{N}}$. In the following, we do not distinguish between a pair (π, ξ) satisfying these conditions and the unique corresponding partial play. We adopt the same point of view for (infinite) plays.

The winning condition $\widehat{\Omega}$ is defined by

$$\begin{aligned}\widehat{\Omega} = \{&(\lambda, \xi) \mid \lambda \in \Omega\}\\ &\cup \{(\lambda, \xi) \mid \exists^{<\infty} \pi v \sqsubset \lambda, \pi \in \mathrm{Dom}(\xi) \text{ and } v \neq \xi(\pi)\}\end{aligned}$$

Remark 2. *As Ω is assumed to be a Borel subset of plays in $\mathbb{G}$, $\widehat{\Omega}$ is a Borel subset of the set of plays in $\widehat{\mathbb{G}}$. Indeed the second part of the condition (which does not involve Ω) is Borel. As the first part is the inverse image of Ω under the* continuous *mapping $\hat{\lambda} \mapsto [\![\hat{\lambda}]\!]$, it is also Borel. Using Borel determinacy [10] the game $\widehat{\mathbb{G}}$ is determined,* i.e. *either Éloïse or Abélard has a winning strategy in $\widehat{\mathbb{G}}$. Furthermore if Ω is ω-regular then so is $\widehat{\Omega}$.*

Theorem 1. *Let $\mathbb{G}$ be a game. The leaking value in $\mathbb{G}$ is at most $\aleph_0$ if and only if Éloïse has a winning strategy in $\widehat{\mathbb{G}}$.*

More precisely, from a winning strategy (resp. positional winning strategy) $\widehat{\varphi}_{\mathrm{E}}$ of Éloïse in $\widehat{\mathbb{G}}$, we can define a strategy (resp. positional winning strategy) φ_{E} for Éloïse in $\mathbb{G}$ such that $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$.

Moreover, from a winning strategy (resp. positional winning strategy) $\widehat{\varphi}_{\mathrm{A}}$ for Abélard, we can define a strategy (resp. positional strategy) φ_{A} for Abélard in $\mathbb{G}$ such that for any *strategy φ_E of Éloïse $Card(\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}} \setminus \Omega) = 2^{\aleph_0}$.*

Proof: First assume that Éloïse has a winning strategy $\widehat{\varphi}_{\mathrm{E}}$ in $\widehat{\mathbb{G}}$. We define a strategy φ_{E} for her in $\mathbb{G}$ as follows. For any partial play π in $\mathbb{G}$ ending in V_{E}, if there exists a partial play of the form (π, ξ) in $\widehat{\mathbb{G}}$ in which Éloïse respects $\widehat{\varphi}_{\mathrm{E}}$ then this play is unique and we take $\varphi_{\mathrm{E}}(\pi) = \widehat{\varphi}_{\mathrm{E}}((\pi, \xi))$. Otherwise $\varphi_{\mathrm{E}}(\pi)$ is undefined.

A straightforward induction shows that for all partial play π ending in V_{E} where Éloïse respects φ_{E} the strategy φ_{E} is defined. Furthermore remark that if $\widehat{\varphi}_{\mathrm{E}}$ is positional, φ_{E} is also positional.

Let us now prove that $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$. For this, fix a strategy φ_{A} of Abélard in $\mathbb{G}$ and consider a play λ in $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}} \setminus \Omega$. As Éloïse respects φ_{E} in λ, there exists by construction of φ_{E}, a unique play of the form (λ, ξ_λ) in $\widehat{\mathbb{G}}$ where Éloïse respects $\widehat{\varphi}_{\mathrm{E}}$. As $\widehat{\varphi}_{\mathrm{E}}$ is winning in $\widehat{\mathbb{G}}$, the corresponding play (λ, ξ_λ) is won by Éloïse and this can only be because Abélard obeys Éloïse only finitely often. Let π_λ be the longest prefix of λ of the form πv with $\pi \in \mathrm{Dom}(\xi_\lambda)$ and $v \neq \xi_\lambda(\pi)$ (*i.e.* π_λ is the last time where Abélard obeys Éloïse).

We claim that $\lambda \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}} \setminus \Omega$ is uniquely characterised by π_λ. In particular $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}} \setminus \Omega$ is countable as it can be injectively mapped into the countable set V^*.

Let $\lambda_1 \neq \lambda_2 \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}} \setminus \Omega$ and let (λ_1, ξ_1) and (λ_2, ξ_2) be the corresponding plays in $\widehat{\mathbb{G}}$. We will show that $\pi_{\lambda_1} \neq \pi_{\lambda_2}$. Consider the greatest common prefix π of λ_1 and λ_2. In particular there exists $v_1 \neq v_2 \in V$ such that $\pi v_1 \sqsubset \lambda_1$ and $\pi v_2 \sqsubset \lambda_2$. As λ_1 and λ_2 respects the same strategies for Éloïse and Abélard, π must end in V_{N}. Moreover for all prefixes of π (including π), ξ_{λ_1} and ξ_{λ_2} coincide. Let $w = \xi_{\lambda_1}(\pi) = \xi_{\lambda_2}(\pi)$ be the vertices Éloïse wants to avoid in π. Assume w.l.o.g. that $w \neq v_1$. Abélard obeys Éloïse at π in (λ_1, ξ_1). In particular, $\pi v_1 \sqsubseteq \pi_{\lambda_1}$ and therefore $\pi_{\lambda_1} \not\sqsubseteq \pi_{\lambda_2}$.

Conversely, assume that Éloïse has no winning strategy in $\widehat{\mathbb{G}}$. By Remark 2, Abélard has a winning strategy $\widehat{\varphi}_{\mathrm{A}}$ in $\widehat{\mathbb{G}}$.

Using $\widehat{\varphi}_{\mathrm{A}}$ we define a strategy φ_{A} of Abélard in $\mathbb{G}$ that is only partialy defined. It can be turned into a full strategy by picking an arbitrary move for Abélard for all partial plays where it is not defined. This transformation can only increase the set of loosing plays for Éloïse and hence we can work with φ_{A} as is.

The strategy φ_{A} uses as a memory a partial play in $\widehat{\mathbb{G}}$, *i.e.* with any partial play π in $\mathbb{G}$ where Abélard respects φ_{A} we associate a partial play $\tau(\pi) = (\pi, \xi)$ in $\widehat{\mathbb{G}}$ where Abélard respects $\widehat{\varphi}_{\mathrm{A}}$. The definition of both $\widehat{\varphi}_{\mathrm{A}}$ and τ are done by induction.

Initially when $\pi = v_0$ one lets $\tau(\pi) = (v_0, \xi)$ where ξ is defined nowhere. Now assume the current partial play is π and that it ends in some vertex v and assume that $\tau(\pi) = (\pi, \xi)$.

- If $v \in V_{\mathrm{A}}$ then $\varphi_{\mathrm{A}}(\pi) = \widehat{\varphi}_{\mathrm{A}}((\pi, \xi)) = v'$ and $\tau(\pi \cdot v') = (\pi \cdot v', \xi)$.
- If $v \in V_{\mathrm{E}}$ and Éloïse moves to v' then $\tau(\pi \cdot v') = (\pi \cdot v', \xi)$.
- If $v \in V_{\mathrm{N}}$ and Nature moves to v' then $\tau(\pi \cdot v')$ is defined only if there exists at least one $w \in E(v)$ such that[3] $\widehat{\varphi}_{\mathrm{A}}(\pi, \xi[\pi \mapsto w]) = v'$. In this case, if $\widehat{\varphi}_{\mathrm{A}}(\pi, \xi[\pi \mapsto v']) = v'$ then we take $\tau(\pi \cdot v') = (\pi \cdot v', \xi[\pi \mapsto v'])$.

[3]We denote $\xi[\pi \mapsto w]$ the extension of ξ where π is mapped to w.

Otherwise we pick $w \in E(v)$ such that $\widehat{\varphi}_{\mathrm{A}}(\pi, \xi[\pi \mapsto w]) = v'$ and set $\tau(\pi \cdot v') = (\pi \cdot v', \xi[\pi \mapsto w])$.

In the last case, remark that $\tau(\pi \cdot v')$ is always defined for at least one $v' \in E(v)$. Furthermore if it is defined for exactly one $v' \in E(v)$, then it is equal to some to $(\pi \cdot v', \xi)$ with $\xi(\pi) = v'$. This means that Abélard disobeys Éloïse.

Finally remark that if $\widehat{\varphi}_{\mathrm{A}}$ is positional, φ_{A} is also positional.

Let φ_{E} be a strategy for Éloïse in $\mathbb{G}$. In order to prove that $CardLeak(\varphi_{\mathrm{E}}) = 2^{\aleph_0}$ we will establish that $Card(\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \backslash \Omega) = 2^{\aleph_0}$.

First remark[4] that $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \cap \Omega = \varnothing$. Indeed, consider a play $\lambda \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$. By construction of φ_{A}, there exists a play of the form (λ, ξ) in $\widehat{\mathbb{G}}$ where Abélard respects $\widehat{\varphi}_{\mathrm{A}}$: in particular it implies that $\lambda \notin \Omega$.

It remains to show that $Card(\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}) \geqslant 2^{\aleph_0}$. Consider the tree $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ of all partial plays respecting both φ_{E} and φ_{A}. To show that $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ has $2^{\aleph_0}$ branches, it is enough to show that every infinite branch in $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ goes through infinitely many nodes with at least 2 successors.

Let λ be a branch in $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ and let $\tau(\lambda) = (\lambda, \xi)$ be the corresponding play in $\widehat{\mathbb{G}}$. As $\tau(\lambda)$ is won by Abélard, he obeys Éloïse infinitely often during this play. Hence there exists $\pi_1 v_1 \sqsubset \pi_2 v_2 \sqsubset \cdots \sqsubset \lambda$ such that for all $i \geqslant 1$, π_i ends in V_N and $\xi(\pi_i) \neq v_i$. As remarked previously for all $i \geqslant 0$, π_i has at least two successors in $T_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}$ (as otherwise it would imply that Abélard disobeys Éloïse at π_i in $\tau(\lambda)$). ■

Remark 3. *One should think of the last part of the statement of Theorem 1 as a determinacy result in the spirit Borel determinacy [10]. Indeed, it states that if Éloïse does not have a strategy that is good against every strategy of Abélard then he has one that is bad (for her) against any of her strategies.*

E. A Game to Decide the Existence of a Topologically Good Strategy

Our goal in this section is to design a technique to decide whether Éloïse has a topological good strategy in a perfect-information game with Nature. We only have result in the case of games where Abélard is not playing (*i.e.* one-player game with Nature).

We start by giving a useful characterisation of large sets of branches in a tree. For this fix a D-tree t for some set of directions D. Call a set of nodes $W \subseteq t$ ***dense*** if $\forall u \in t$, $\exists v \in W$ such that $u \sqsubseteq v$. Given a dense set of nodes W, the set of branches supported by W, $\mathcal{B}(W)$ is the set of branches π that have infinitely many prefixes in W. Using the existence of decomposition-invariant winning strategies in Banach-Mazur games, the following lemma from [9] characterises large sets of branches.

Lemma 1. *Let t be a D-tree for some D and B be a Borel set of branches in t. Then B is large if and only if there exists a dense set of nodes $W \subseteq t$ such that $\mathcal{B}(W) \subseteq B$.*

In order to describe a dense set of nodes, we mark a path to this set in the tree as follows. Let t be a tree. A ***direction mapping*** is a mapping $d : t \to D$, and given a set of nodes W, we say that d ***points to W*** if for every node u there exists $d_1, \ldots, d_k \in D$ such that $ud_1 \cdots d_k \in W$ and for all $1 \leqslant j \leqslant k$, $d_j = d(ud_1 \cdots d_{j-1})$.

Lemma 2. *A set of nodes W is dense if and only if there exists a direction mapping that points to W.*

Fix an ***arena*** $\mathcal{G} = (G = (V, E), V_{\mathrm{E}}, V_{\mathrm{A}}, V_{\mathrm{N}})$ where we have $V_{\mathrm{A}} = \varnothing$ and a game $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ (*i.e.* Abélard is not part of the game). We assume that the game is turn based, *i.e.* that $E \subseteq V_{\mathrm{E}} \times V_{\mathrm{N}} \cup V_{\mathrm{N}} \times V_{\mathrm{E}}$, and that $v_0 \in V_{\mathrm{E}}$. This restriction is not essential but highly simplifies the presentation.

We design a two-player perfect information game without Nature such that Éloïse wins in $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}, v_0, \widetilde{\Omega})$ if and only if she has a topologically good strategy in $\mathbb{G}$.

The arena $\widetilde{\mathcal{G}}$ of the game $\widetilde{\mathbb{G}}$ is quite similar to $\mathcal{G}$ and the main intuition is that Éloïse mimics a play against Nature in $\mathbb{G}$ and additionally describe a dense set of nodes W (thanks to a direction mapping and an explicit annotation of nodes in W) in the tree of possible outcomes. Abélard simulates the moves of Nature and he tries either to prove that W is not dense or that there is a loosing play in $\mathcal{B}(W)$. Formally one defines $\widetilde{\mathcal{G}} = (\widetilde{V} = \widetilde{V}_{\mathrm{E}} \cup \widetilde{V}_{\mathrm{A}}, \widetilde{E})$ where $\widetilde{V}_{\mathrm{E}} = V_{\mathrm{E}}$, $\widetilde{V}_{\mathrm{A}} = V_{\mathrm{N}} \times V_{\mathrm{E}} \times \{\top, \bot\}$ and

$$\begin{aligned}\widetilde{E} = \{(v, (v', w, b)) \mid (v, v') \in E \cap V_{\mathrm{E}} \times V_{\mathrm{N}},\ w \in E(v') \text{ and} \\ b \in \{\top, \bot\}\} \cup \{((v, w, b), v') \mid (v, v') \in E \cap V_{\mathrm{N}} \times V_{\mathrm{E}}\}\end{aligned}$$

Intuitively in a partial play λ, by choosing an edge from v to (v', w, b) Éloïse indicates that the direction mapping in $\lambda \cdot v'$ is to go to w; moreover if $b = \top$ she indicates that $\lambda \cdot v'$ is in the dense set W (remark that, due to the turn base nature of the game, one can safely assume that the element in W are always partial plays ending in a vertex in V_{N}). A play is winning for Éloïse if either it satisfies the winning condition while visiting infinitely many nodes marked as belonging to the dense set or if at some point no more position in W are reached while Abélard infinitely often selects a direction that is not the one given by the direction mapping (*i.e.* he does not let Éloïse a chance to get to a position in W).

$$\begin{aligned}\widetilde{\Omega} = \{v_0(v_0', w_0, b_0)v_1(v_1', w_1, b_1)v_2 \cdots \mid v_0 v_0' v_1 v_1' v_2 v_2' \cdots \in \Omega \\ \text{and } \exists^{\infty} j \text{ s.t. } b_j = \top\} \\ \cup \{v_0(v_0', w_0, b_0)v_1(v_1', w_1, b_1)v_2 \cdots \mid \exists^{<\infty} j \text{ s.t. } b_j = \top \\ \text{and } \exists^{\infty} j \text{ s.t. } v_{j+1} \neq w_j)\}\end{aligned}$$

The following result connects the games $\mathbb{G}$ and $\widetilde{\mathbb{G}}$.

Theorem 2. *Éloïse has a topologically good strategy in $\mathbb{G}$ if and only if she has a winning strategy in $\widetilde{\mathbb{G}}$.*

More precisely, from a winning strategy (resp. positional winning strategy) $\widetilde{\varphi}_{\mathrm{E}}$ of Éloïse in $\widetilde{\mathbb{G}}$, we can define a topologically good strategy (resp. positional strategy) φ_{E} for Éloïse in $\mathbb{G}$.

Proof: Assume that Éloïse has a topologically good strategy in $\mathbb{G}$. Call φ this strategy and let t_{φ} be the set of

[4]This is no longer true for the full version of φ_{A}

all partial plays starting from v_0 where Éloïse respects φ. By definition t_φ is a tree and its branches are those plays in $\mathbb{G}$ where Éloïse respects φ. As φ is topologically good the set of branches in t_φ that belongs to Ω is large and therefore thanks to Lemma 1 it contains a dense set of nodes $\Lambda \subseteq V_{\mathrm{N}}$ that, using Lemma 2, can be described by a direction mapping d.

Define a strategy $\widetilde{\varphi}_{\mathrm{E}}$ in $\widetilde{\mathbb{G}}$ for Éloïse by letting $\widetilde{\varphi}_{\mathrm{E}}(v_0(v_0', w_0, b_0)v_1(v_1', w_1, b_1)\dots v_k) = (v_k', w_k, b_k)$ where $v_k' = \varphi(v_0 v_0' v_1 v_1' \dots v_k)$, $w_k = d(v_0 v_0 v_1 v_1' \dots v_k v_k')$ and $b_k = \top$ if $v_0 v_0' v_1 v_1' \dots v_k v_k' \in \Lambda$ and $b_k = \bot$ otherwise.

Now consider a play $\widetilde{\lambda} = v_0(v_0', w_0, b_0)v_1(v_1', w_1, b_1)\dots$ in $\widetilde{\mathbb{G}}$ where Éloïse respects $\widetilde{\varphi}_{\mathrm{E}}$: if it goes infinitely often through vertices in $V_{\mathrm{N}} \times V_{\mathrm{E}} \times \{\top\}$ then $v_0 v_0' v_1 v_1' \dots$ is an infinite branch in t_φ that goes through infinitely many nodes in Λ hence, belongs to Ω and so $\widetilde{\lambda} \in \widetilde{\Omega}$; otherwise, thanks to the direction mapping and the definition of $\widetilde{\varphi}_{\mathrm{E}}$ it follows that if eventually Abélard always chooses to go from (v', w, b) to w then one eventually reaches a vertex in $V_{\mathrm{N}} \times V_{\mathrm{E}} \times \{\top\}$ and therefore $\widetilde{\lambda} \in \widetilde{\Omega}$.

Conversely, assume that Éloïse has a winning strategy $\widetilde{\varphi}_{\mathrm{E}}$ in $\widetilde{\mathbb{G}}$. We define a strategy φ for Éloïse in $\mathbb{G}$ as follows. The strategy φ is defined so that with a partial play λ in $\mathbb{G}$ (where she respects φ) is associated a partial play $\widetilde{\lambda}$ in $\widetilde{\mathbb{G}}$ (where she respects $\widetilde{\varphi}$). Initially $\lambda = \widetilde{\lambda} = v_0$. Let $\lambda = v_0 v_0' v_1 v_1' \cdots v_k$ be a partial play where she respects φ and let $\widetilde{\lambda} = v_0(v_0', w_0, b_0)v_1(v_1', w_1, b_1)\cdots v_k$; then call $\widetilde{\varphi}(\widetilde{\lambda}) = (v_k', w_k', b_k)$; define $\varphi(\lambda) = v_k'$ and $\widetilde{\lambda v_k'} = \widetilde{\lambda}(v_k', w_k', b_k)$. Now let t_φ be the set of all partial plays starting from v_0 where Éloïse respects φ. Define the set of nodes Λ in t_φ of those partial plays that ends in V_{N} and such that $\widetilde{\lambda}$ ends in a vertex in $V_{\mathrm{N}} \times \mathbb{N} \times \{\top\}$ and define a direction mapping d in t_φ by letting, for any λ ending in V_{N}, $d(\lambda) = w$ where w is such that $\widetilde{\lambda}$ ends in a vertex in $V_{\mathrm{N}} \times \{w\} \times \{\bot, \top\}$ (in other nodes there is a single son so there is only one way to define d). As $\widetilde{\varphi}_{\mathrm{E}}$ is winning one easily deduces that d is a direction mapping that points to Λ and that $\mathcal{B}(\Lambda) \subseteq \Omega$. Therefore, the subset of branches of t_φ that satisfies Ω is large, meaning that φ is topologically good. ∎

IV. Imperfect-Information Games with Nature

We now move to a richer setting where Éloïse has imperfect information. The vertices of the game are partitioned by an equivalence relation and Éloïse does not observe exactly the current vertex but only its equivalence class. In full generality, Abélard should also have imperfect information but we assume here that he is perfectly informed. Of course, as Éloïse has imperfect information we have to slightly change the definition of the game (she now plays actions) and to restrict the strategies she can use. We also change how Nature interacts with the players, but one can easily check that this setting captures the one we gave in the perfect-information case[5].

[5]One could wonder why we did not directly treat the imperfect information case. There are two main reasons for that. Firstly, in the imperfect information setting we only have results for the parity condition and not for any Borel condition. Secondly, the proof of Theorem 3 crucially uses the results obtained in the perfect information setting.

A. Definitions

An ***imperfect-information arena*** is a tuple $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ where V_{E} is a countable set of Éloïse's vertices, V_{A} is a countable set of Abélard's vertices (we let $V = V_{\mathrm{E}} \uplus V_{\mathrm{A}}$), Γ is a possibly uncountable set of Éloïse's actions, $\Delta_{\mathrm{E}} : V_{\mathrm{E}} \times \Gamma \to 2^V$ is Éloïse's transition function and $\Delta_{\mathrm{A}} : V_{\mathrm{A}} \to 2^V$ is Abélard's transition function and $\sim$ is an equivalence relation on V. We additionally require that the image by Δ_{E} (resp. Δ_{A}) is never the empty set. We also require that there is no vertex $v_1 \in V_{\mathrm{E}}$ and $v_2 \in V_{\mathrm{A}}$ such that $v_1 \sim v_2$ (*i.e.* the $\sim$ relation distinguishes between vertices from different players).

Again, a play involves two antagonistic players — Éloïse and Abélard — together with an unpredictable and uncontrollable player called Nature. The play starts in some initial vertex v_0 and when in some vertex v the following happens:

- if $v \in V_{\mathrm{E}}$, Éloïse chooses an action γ and then Nature chooses the next vertex $v' \in \Delta_{\mathrm{E}}(v, \gamma)$;
- if $v \in V_{\mathrm{A}}$, Abélard chooses the next vertex $v' \in \Delta_{\mathrm{A}}(v)$.

Then, the play goes on from v' and so on forever.

Hence, a ***play*** can be seen as an element in $(V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^\omega$ compatible with Δ_{E} and Δ_{A}. A ***partial play*** is a prefix of a play, *i.e.* it belongs to $(V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^*$.

Two $\sim$-equivalent vertices are supposed to be indistinguishable by Éloïse and we extend $\sim$ as an equivalence relation on V^*: $v_0 \dots v_h \sim v_0' \dots v_k'$ if and only if $h = k$ and $v_i \sim v_i'$ for all $0 \leqslant i \leqslant k$; we denote by $[\pi]/_\sim$ the equivalence class of $\pi \in V^*$. An ***observation-based strategy*** for Éloïse is a map $\varphi : V^*/_\sim \cdot V_{\mathrm{E}\sim} \to \Gamma$. We say that Éloïse ***respects*** φ in play $\lambda = v_0\gamma_0 v_1 \gamma_1 v_2 \gamma_2 \cdots$ (where γ_i is the empty word when $v_i \in V_{\mathrm{A}}$ and an action in Γ when $v_i \in V_{\mathrm{E}}$) if and only if $\gamma_{i+1} = \varphi([v_0 \cdots v_i]/_\sim)$ for all $i \geqslant 0$ such that $v_i \in V_{\mathrm{E}}$.

Remark 4. *One may expect a strategy for Éloïse to depend also on the actions played so far,* i.e. *to be a map* $\varphi_{\mathrm{E}} : (V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^* \cdot V_{\mathrm{E}} \to \Gamma$. *But such a strategy can be mimicked by a strategy (in our sense)* $\varphi_{\mathrm{E}}' : V^* \to \Gamma$ *by letting* $\varphi_{\mathrm{E}}'(v_0 \cdots v_k) = \varphi_{\mathrm{E}}(v_0\gamma_0 \cdots \gamma_{k-1} v_k)$ *with* $\gamma_i = \varphi_{\mathrm{E}}(v_0\gamma_0 \cdots \gamma_{i-1} v_i)$ *when* $v_k \in V_{\mathrm{E}}$ *and* $\gamma_i = \varepsilon$ *otherwise. Note that requiring to be observation-based does not interfere with the previous trick.*

A ***strategy*** for Abélard is a map $\varphi : (V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^* \cdot V_{\mathrm{A}} \to V$. We say that Abélard ***respects*** φ in the play $\lambda = v_0\gamma_0 v_1 \gamma_1 v_2 \gamma_2 \cdots$ (again, γ_i is the empty word when $v_i \in V_{\mathrm{A}}$ and an action in Γ when $v_i \in V_{\mathrm{E}}$) if and only if $v_{i+1} = \varphi(v_0 \gamma_0 v_1 \cdots v_i)$ for all $i \geqslant 0$ such that $v_i \in V_{\mathrm{A}}$.

With an initial vertex v_0, a strategy φ_{E} of Éloïse and a strategy φ_{A} of Abélard, we associate the set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}}$ of possible plays starting from v_0 and where Éloïse (resp. Abélard) respects φ_{E} (resp. φ_{A}).

In this part, we only have positive results for parity winning conditions, hence we focus on this setting (but generalising the notions to any Borel winning condition is straightforward). A parity winning condition is given thanks to a colouring function $\mathrm{Col} : V \to C$ with a *finite* set of colours $C \subset \mathbb{N}$.

Again, a play $\lambda = v_0\gamma_0 v_1\gamma_1 v_2\gamma_2\cdots$ satisfies the parity condition if $\liminf(\mathrm{Col}(v_i))_{i\geqslant 0}$ is even; we denote by Ω_{Col} the set of plays satisfying the parity condition defined by the colouring function Col.

A ***imperfect-information parity game with nature*** is a tuple $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$ consisting of an imperfect-information arena $\mathcal{G}$, a colouring function Col and an initial vertex v_0.

Remark 5. *A more natural notion of imperfect-information game would have Abélard also playing actions (*i.e. $\Delta_{\mathrm{A}} : V_{\mathrm{A}} \times \Gamma \mapsto 2^V$*) and Nature would choose the successor as it does for Éloïse. Consider such a game* $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ *where Abélard plays actions. We can simulate it by a game* $\mathcal{G} = (V'_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta'_{\mathrm{E}}, \Delta'_{\mathrm{A}}, \sim')$ *in our setting. For every vertex* v *of Abélard and every action* $\gamma \in \Gamma$*, we introduce a new vertex* (v,γ) *for Éloïse (*i.e. $V'_{\mathrm{E}} = V_{\mathrm{E}} \cup V_{\mathrm{A}} \times \Gamma$*). Furthermore we set* $\Delta'_{\mathrm{A}}(v) = \{(v,\gamma) \mid \gamma \in \Gamma\}$ *and for all vertex of Éloïse of the form* (v,γ)*, we take* $\Delta'_{\mathrm{E}}((v,\gamma),\gamma') = \Delta_{\mathrm{A}}(v,\gamma)$ *for all action* $\gamma' \in \Gamma$*. For the original vertices* $v \in V_{\mathrm{E}}$ *and for* $\gamma \in \Gamma$*, we take* $\Delta'_{\mathrm{E}}(v,\gamma) = \Delta_{\mathrm{E}}(v,\gamma)$*. Finally the equivalence relation* $\sim'$ *coincides with* $\sim$ *and equates all new vertices.*

In order to evaluate how good a strategy for Éloïse is, we can take the same definitions as we did in the perfect information setting. Hence, we have the notions of ***cardinality leaking*** of a strategy (thanks to Definition 1), ***leaking value*** of a game (thanks to Definition 2)[6], and ***topologically good*** strategy (thanks to Definition 3),

We now introduce another version of games with imperfect information where there are only two antagonist players — Éloïse and Abélard. The only difference with the previous model with Nature is that now the non-determinism induced by a choice of an action of Éloïse is resolved by Abélard. This concept was first considered in [15] for finite arenas.

Let $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ be an imperfect-information arena. Then a play involves two players Éloïse and Abélard: it starts in some initial vertex v_0 and when in some vertex v the following happens:

- if $v \in V_{\mathrm{E}}$, Éloïse chooses an action γ and then Abélard chooses the next vertex $v' \in \Delta_{\mathrm{E}}(v,\gamma)$;
- if $v \in V_{\mathrm{A}}$, Abélard chooses the next vertex $v' \in \Delta_{\mathrm{A}}(v)$.

Then, the play goes on from v' and so on forever. Again a ***play*** is as an element in $(V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^\omega$ and a ***partial play*** is one in $(V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^*$.

Observation-based strategies for Éloïse are defined as for imperfect-information games with Nature. We shall consider winning conditions slightly more general than parity conditions hence, we allow any Borel subset Ω of $(V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^\omega$.

An ***imperfect-information two-player game*** is a tuple $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ consisting of an arena of imperfect-information, a winning condition Ω and an initial vertex v_0. A strategy φ_{E} of Éloïse is ***winning*** in $\mathbb{G}$ if any play starting from v_0 where Éloïse respects φ_{E} belongs to Ω.

[6]For the same reason as in the prefect-information setting we have that for any strategy φ_{E} $CardLeak(\varphi) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ and as a consequence that $LeakVal(\mathbb{G}) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Remark 6. *Note that even for reachability conditions —* i.e. $\Omega = V^*FV^\omega$ *for some non-empty* $F \subseteq V$ *— and finite arena, imperfect-information two-player games are not determined,* i.e. *it can happen that none of the player as a winning strategy. See [15, Example 2.3].*

B. Deciding Whether the Leaking Value Is at Most $\aleph_0$

Our goal in this section is to design a technique to decide whether Éloïse has a strategy with a cardinality leaking of at most $\aleph_0$ in an imperfect-information parity game with nature.

For the rest of this section we fix an imperfect-information parity game with nature $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$ where $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ and we aim at deciding whether $LeakVal(\mathbb{G}) \leqslant \aleph_0$.

The approach is similar to the perfect information case. We define an imperfect-information game without Nature where Abélard is now in charge of simulating choices of Nature while Éloïse will indicate together with her action, a successor that she wants to avoid. Moreover Abélard will be forced (thanks to the winning condition) to respect her choices infinitely often.

In order to express the choice of Nature she wants to avoid while preserving the fact that she is partially informed about the actual vertex, Éloïse will provide with her action $\gamma \in \Gamma$, a map $\theta : V \to V$ such that for all $v \in V$ one has $\theta(v) \in \Delta_{\mathrm{E}}(v,\gamma)$; we denote by Θ_γ the set of such maps (for a given $\gamma \in \Gamma$). Intuitively the meaning of Éloïse playing (γ,θ) is that she plays action γ and would prefer that if the play is in some vertex v that it avoids $\theta(v)$.

Remark 7. *The map* θ *may be partial: what is important is that, if at some point the play can be in* v *then* $\theta(v)$ *should be defined. In particular if there are two bounds, one on the size of the equivalence classes of* $V_{/\sim}$ *and one on the out-degree of the vertices in* G*, then* Θ_γ *can be chosen to be finite (up to coding). This will be the case for pushdown games when discussing consequences in Section V-B.*

We define a two-player imperfect-information arena $\widehat{\mathcal{G}} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \widehat{\Gamma}, \widehat{\Delta}_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ where $\widehat{\Gamma} = \bigcup_{\gamma\in\Gamma}\{\gamma\} \times \Theta_\gamma$ and $\widehat{\Delta}_{\mathrm{E}}(v,(\gamma,\theta)) = \Delta_{\mathrm{E}}(v,\gamma)$. A play on $\widehat{\mathcal{G}}$ is of the form $v_0\nu_0 v_1\nu_1 v_2\cdots$ where for all $i \geqslant 0$, ν_i is equal to (γ_i,θ_i) if $v_i \in V_{\mathrm{E}}$ and is empty otherwise. For some $i \geqslant 0$, if $v_i \in V_{\mathrm{E}}$ and $v_{i+1} \neq \theta_i(v_i)$, we say that Abélard *obeys* Éloïse at this point.

We let $\widehat{\Omega}$ consists of those plays $v_0\nu_0 v_1\nu_1 v_2\cdots$ such that either $v_0 v_1 v_2 \cdots \in \Omega_{\mathrm{Col}}$ or there are only finitely many i such that $v_i \in V_{\mathrm{E}}$ and $v_{i+1} \neq \theta_i(v_i)$, *i.e.* either the play satisfies the parity condition or eventually Abélard never obeys Éloïse. Finally we denote by $\widehat{\mathbb{G}}$ the two-player imperfect-information game $(\widehat{\mathcal{G}}, \widehat{\Omega}, v_0)$. The next result relates $\mathbb{G}$ and $\widehat{\mathbb{G}}$.

Theorem 3. *The leaking value of* $\mathbb{G}$ *is at most* $\aleph_0$ *if and only if Éloïse has a winning strategy in* $\widehat{\mathbb{G}}$.

More precisely, from a winning strategy (resp. positional winning strategy) $\widehat{\varphi}_{\mathrm{E}}$ *of Éloïse in* $\widehat{\mathbb{G}}$*, we can define a strategy (resp. positional winning strategy)* φ_{E} *for Éloïse in* $\mathbb{G}$ *such that* $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$.

Proof: Let π be a partial play in $\mathbb{G}$ (resp. $\widehat{\mathbb{G}}$), we denote by $[\![\pi]\!]$ the sequence of vertices in V^* obtained by removing the actions from π. For any partial play π in $\mathbb{G}$ in which Éloïse respects φ_{E}, there exists a unique play, denoted $\hat{\pi}$, in which Éloïse respects $\widehat{\varphi}_{\mathrm{E}}$ and such that $[\![\pi]\!] = [\![\hat{\pi}]\!]$. By taking the limit, we extend this notation from partial plays to plays.

First assume that Éloïse has a winning strategy $\widehat{\varphi}_{\mathrm{E}}$ in $\widehat{\mathbb{G}}$ and define a strategy φ_{E} for Éloïse in $\mathbb{G}$ by letting $\varphi_{\mathrm{E}}(\lambda) = \gamma$ whenever $(\gamma, \theta) = \widehat{\varphi}_{\mathrm{E}}(\lambda)$. Using the same ideas as in the perfect information case we prove that $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$.

For the converse implication, we cannot proceed as in the perfect information case as the game $\widehat{\mathbb{G}}$ is not determined. Assume that the leaking value of $\mathbb{G}$ is at most $\aleph_0$ and let φ_{E} be a strategy of Éloïse such that $LeakVal(\varphi_{\mathrm{E}}) \leqslant \aleph_0$ (thanks to Remark 1 it exists).

In order to define a winning strategy in $\widehat{\mathbb{G}}$ for Éloïse, we consider a perfect information parity game with Nature $\mathbb{P}$. This game is only played between Abélard and Nature.

To define this game, consider the set S as the set of all $[\![\pi]\!]$ for π a partial play respecting φ_{E} and the equivalence relation $\equiv$ on S defined for all η,$\eta' \in S$ by $\eta \equiv \eta'$ if η and η' end in the same vertex and $\eta \sim \eta'$.

The vertices of this game $V_{\mathbb{P}}$ are the equivalence classes $\equiv$. A vertex $[\eta]_{/\equiv} \in V_{\mathbb{P}}$ belongs to Abélard if η ends in a vertex of Abélard and it belongs to Nature otherwise. There is an edge from $[\eta]_{/\equiv}$ to $[\eta']_{/\equiv}$ if η' extends η by one vertex. The initial vertex is $[v_0]_{/\equiv}$ and the parity condition is given by the mapping associating to $[\eta]_{/\equiv} \in V_{\mathbb{P}}$ the colour $\mathrm{Col}(v)$ of the the last vertex v of η.

A partial play ξ in $\mathbb{P}$ is of the form $[\eta_0]_{/\equiv}[\eta_1]_{/\equiv} \cdots [\eta_n]_{/\equiv}$ where $\eta_0 = v_0$ and for all $i < n$, η_{i+1} extends η_i by one vertex. We naturally associate the play $\tau(\xi)$ in $\mathbb{G}$ defined by $v_0\nu_0 v_1 \nu_1 \cdots v_n$ where for all $i \geqslant 0$, η_i ends in v_i and ν_i is equal to $\varphi_{\mathrm{E}}([\eta_i]_{/\sim})$ if v_i belongs to Éloïse and ν_i is empty otherwise. It is easy to show that for all $i \leqslant n$, $v_0 v_1 \cdots v_i \equiv \eta_i$. Hence as φ_{E} is observation-based, $\tau(\xi)$ respects φ_{E}. In fact, the continuous mapping τ establishes a one to one correspondance between the partial plays in $\mathbb{P}$ and the partial plays in $\mathbb{G}$ which respect φ_{E}. By continuity, this mapping extends to plays.

The game $\mathbb{P}$ is won by Éloïse. Indeed any strategy $\varphi^{\mathbb{P}}_{\mathrm{A}}$ for Abélard in $\mathbb{P}$ can be lifted to a strategy φ_{A} in $\mathbb{G}$ such that $\{\tau(\xi) \mid \xi \text{ a play in } \mathbb{P} \text{ which respects } \varphi^{\mathbb{P}}_{\mathrm{A}}\}$ is equal to $\mathrm{Outcomes}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}}_{v_0,\mathbb{G}}$.

By Theorem 1, Éloïse has a winning strategy in the game $\widehat{\mathbb{P}}$. As the winning condition of $\mathbb{P}$ is a disjunction of parity conditions, the winning condition of $\widehat{\mathbb{P}}$ is a Rabin condition[7]. Therefore Éloïse has a positional winning strategy $\varphi^{\widehat{\mathbb{P}}}_{\mathrm{E}}$ in $\widehat{\mathbb{P}}$ [16]. For $\eta \in S$ ending with a vertex of Éloïse, $\varphi^{\widehat{\mathbb{P}}}_{\mathrm{E}}$ associates to $[\eta]_{/\equiv}$ a pair $([\eta]_{/\equiv}, [\eta v']_{/\equiv})$ with $v' \in \Delta_{\mathrm{E}}(v, \varphi_{\mathrm{E}}([\eta]_{/\sim}))$. This strategy is completely characterised by the mapping φ_B associating to $[\eta]_{/\equiv}$ the vertex v' in $\Delta_{\mathrm{E}}(v, \varphi_E([\eta]_{/\sim}))$.

[7]The Rabin condition is in fact on the sequence of edges taken during the play and not on sequence of vertices visited. By a slight modification of $\widehat{\mathbb{P}}$, it can be transformed into a Rabin condition on the sequence of vertices visited.

The key property of this strategy is that any play λ in $\mathbb{G}$ which respects φ_{E} and such that λ has infinitely many prefixes of the form $\pi v \gamma v'$ with $v \in V_{\mathrm{E}}$ and $v' \neq \varphi_B([[\![\pi v]\!]]_{/\equiv})$, satisfies the parity condition. Indeed, toward a contradiction assume that λ does not satisfy the parity condition. Let $\lambda' = \tau^{-1}(\lambda)$ be the corresponding play in $\mathbb{P}$ and let (λ', φ_B) be the corresponding play in $\widehat{\mathbb{P}}$. None of these plays satisfies the parity condition. However as (λ', φ_B) respects the positional winning strategy for Éloïse described by φ_B, it is won by Éloïse. This implies that Abélard only obeys Éloïse finitely often which brings the contradiction.

In order to define a strategy for Éloïse in $\widehat{\mathbb{G}}$ we will mimic φ_{E} to choose the Γ-component (call γ the action) and use φ_B to choose the Θ_γ-component.

For this we let $\widehat{\varphi}_{\mathrm{E}}([\eta]_{/\sim}) = (\gamma, \theta)$ where $\gamma = \varphi_{\mathrm{E}}([\eta]_{/\sim})$ and θ is defined as follows. Let $v \in V$: if there exists $\eta' \sim \eta$ ending with v we take $\theta(v) = \varphi_B([\eta']_{/\equiv})$; otherwise we define $\theta(v) = w$ for some arbitrary $w \in \Delta(v, \gamma)$ (the value actually does not matter).

Now consider a play $\widehat{\lambda} = v_0\nu_0 v_1 \nu_1 v_2 \cdots$ in $\widehat{\mathbb{G}}$ where Éloïse respects $\widehat{\varphi}_{\mathrm{E}}$, note $\nu_i = (\gamma_i, \theta_i)$ when $\nu_i \neq \varepsilon$ (*i.e.* when $v_i \in V_{\mathrm{E}}$) and define $\gamma_i = \varepsilon$ when $\nu_i = \varepsilon$. By contradiction assume that $\widehat{\lambda}$ is loosing for Éloïse. Consider the play $\lambda = v_0\gamma_0 v_1 \gamma_1 v_2 \cdots$: it is a play in $\mathbb{G}$ where Éloïse respects φ_{E} and as $\widehat{\lambda} \notin \widehat{\Omega}$ one also has $\lambda \notin \Omega_{\mathrm{Col}}$. But as $\widehat{\lambda}$ is loosing for Éloïse it means that for infinitely many i one has $v_{i+1} \neq \theta_i(v_i)$, which implies that for infinitely many i one has $v_{i+1} \neq \varphi_B([v_0 \cdots v_i]_{/\equiv})$. Therefore as remarked previously, it implies that $\lambda \in \Omega_{\mathrm{Col}}$ hence, leading a contradiction. ■

C. Deciding the Existence of a Topologically Good Strategy

Our goal in this section is to design a technique to decide whether Éloïse has a topological good strategy in an imperfect-information parity games with Nature. We only have results in the case of games where Abélard is not playing (*i.e.* one-player game with Nature) hence, we implicitly assume this from now.

We start by giving a useful characterisation of large sets of branches in a tree when the set of branches is defined by a parity condition. For this fix a D-tree t for some set of directions D. Assume that we have a colouring function $\mathrm{Col} : t \to C$ for a finite set C of colours.

Call a ***local-strategy*** for Éloïse a pair (φ_f, φ_n) of two maps from $t \to D \times \{\top, \bot\}$. For all node $u \in t$, we let $d_f(u)$ (resp. $d_n(u)$) be the unique element such that $\varphi_f(u) \in \{d_f(u)\} \times \{\top, \bot\}$ (resp. $\varphi_n(u) \in \{d_n(u)\} \times \{\top, \bot\}$).

A local-strategy is *valid* if the following holds.

1) For every $u \in t$ both $u \cdot d_f(u)$ and $u \cdot d_n(u)$ are nodes in t, *i.e.* φ_f and φ_u indicates an existing son.
2) For every $u \in t$ there is a node $v = ud_1 \cdots d_\ell$ such that $\varphi_f(v) \in D \times \{\top\}$ and $d_i = d_f(ud_1 \cdots d_{i-1})$ for all $i < \ell$; *i.e.* following φ_f leads to a node where the second component is $\top$.
3) For every $u \in t$ there is a node $v = ud_1 \cdots d_\ell$ such that $\varphi_n(v) \in D \times \{\top\}$ and $d_i = d_n(ud_1 \cdots d_{i-1})$ for all

$i < \ell$; *i.e.* following φ_n leads to a node where the second component is $\top$.

Take a valid local-strategy (φ_f, φ_n). A (φ_f, φ_n)-compatible branch is any branch in t that can be obtained as follows: one selects any node u_0 in t and then one lets v_0 be the shortest node satisfying property (2) above (w.r.t. node u_0), then one selects any node u_1 such that $v_0 \sqsubset u_1$ and one lets v_1 be be the shortest node satisfying property (3) above (w.r.t. node u_1), then one selects any node u_2 such that $v_1 \sqsubset u_2$ and one lets v_2 be the shortest node satisfying property (3) above (w.r.t. node u_2), and so on forever (*i.e.* we use property (2) only in the first round and then we use property (3) forever).

We have the following lemma (whose proof follows the one of [13, Proposition 13]).

Lemma 3. *The set of branches satisfying the parity condition in t is large if and only if there is a valid local-strategy (φ_f, φ_n) such that any (φ_f, φ_n)-compatible branch satisfies the parity condition. Moreover one can choose (φ_f, φ_n) such that $\varphi_f(u_1) = \varphi_f(u_2)$ and $\varphi_n(u_1) = \varphi_n(u_2)$ whenever $t[u_1] = t[u_2]$.*

Recall that we assume that Abélard is not part of the game. Hence, we omit him in notations when considering the original game (*i.e.* we do not write V_{A} neither Δ_{A}).

For the rest of this section we fix an imperfect-information one-player parity game with nature $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$ where $\mathcal{G} = (V, \Gamma, \Delta, \sim)$ and we aim at deciding whether Éloïse has a topologically good strategy.

The main idea is to define an imperfect-information game without Nature but with Abélard. In this game Éloïse simulates a play in $\mathbb{G}$ and also describes a local-strategy for a Banach-Mazur game played on the outcomes; Abélard is in charge of simulating the Banach-Mazur game: sometimes he chooses the directions and sometimes he plays what the local-strategy of Éloïse is indicating. Moreover Éloïse does not observe who is currently playing in the Banach-Mazur game. The winning condition checks the parity condition as well as correctness of the simulation of the Banach-Mazur game (in particular that no player plays eventually forever).

In order to describe the local-strategy, Éloïse will provide with any action $\gamma \in \Gamma$ a partial map $\theta : V \to (V \times \{\top, \bot\}) \times (V \times \{\top, \bot\})$ such that for all $v \in V$ one has $\theta(v) \in \Delta(v, \gamma) \times \{\top, \bot\} \times \Delta(v, \gamma) \times \{\top, \bot\}$; we denote by Θ_γ the set of such maps (for a given $\gamma \in \Gamma$).

We define a two-player imperfect-information arena (all vertices belong to Éloïse so we omit vertices and the transition relation of Abélard) $\widetilde{\mathcal{G}} = (\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \approx)$ where $\widetilde{V} = V \times \{E, A\} \times \{f, n\}$ (the second component is used to remember who plays in the simulation of the Banach-Mazur game; the third component is f if the first move of Éloïse in the Banach-Mazur game has not yet been fully played), $(v, X, x) \approx (v', Y, y)$ if and only if $v \sim v'$ (Éloïse does not observe the second and third components), $\widetilde{\Gamma} = \bigcup_{\gamma \in \Gamma} \{\gamma\} \times \Theta_\gamma$ and $\widetilde{\Delta}((v, X, x), (\gamma, \theta))$ is as follows.

- If $X = A$ then it equals $\Delta(v, \gamma) \times \{E, A\} \times \{x\}$: Abélard can choose any successor and can decide to finish/continue his move in the Banach-Mazur component.
- If $X = E$ then it is the singleton consisting of node (v_x, Y, y) defined by letting[8] $\theta(v) = (v_f, y_f, v_n, y_n)$ and letting $Y = A$ and $y = n$ if $y_x = \top$ (we switch the player in the Banach-Mazur game) and $Y = E$ and $y = x$ if $y_x = \bot$ (she keeps playing).

We let $\widetilde{\Omega}$ consists of those plays $(v_0, X_0, x_0)(v_1, X_1, x_1)(v_2, X_2, x_2)\cdots$ such that either (i) $v_0 v_1 v_2 \cdots$ satisfies the winning condition and one has $X_j = A$ for infinitely many j (*i.e.* Éloïse does not eventually play forever in the Banach-Mazur game) or (ii) there is some $N \geqslant 0$ such that one has $X_j = A$ for all $j \geqslant N$ (*i.e.* Abélard eventually plays forever in the Banach-Mazur game) . In particular $\widetilde{\Omega}$ is a (positive) Boolean combination of Ω and a parity condition.

Finally we denote by $\widetilde{\mathbb{G}}$ the imperfect-information game $(\widetilde{\mathcal{G}}, \widetilde{\Omega}, (v_0, A, f))$. The following relates the games $\mathbb{G}$ and $\widetilde{\mathbb{G}}$.

Theorem 4. *Éloïse has a topologically good strategy in $\mathbb{G}$ if and only if she has a winning strategy in $\widetilde{\mathbb{G}}$.*

More precisely, from a winning strategy (resp. positional strategy) $\widetilde{\varphi}_{\mathrm{E}}$ of Éloïse in $\widetilde{\mathbb{G}}$, we can define a topologically good strategy (resp. positional strategy) φ_{E} for Éloïse in $\mathbb{G}$.

V. Consequences

A. Some Consequences in the Perfect-Information Setting

1) The Special Case of Parity Games on Finite Arenas: In the following statement we make no assumption on the probability distribution put on the transitions.

Theorem 5. *Let $\mathbb{G}$ be a game with an ω-regular winning condition played on a finite arena by only Éloïse and Nature. Then Éloïse has an almost-surely winning strategy if and only if she has a topologically good strategy.*

Proof: If there exists an almost surely winning strategy it is well known that it can be chosen with finite memory, *i.e.* realised by a finite transducer, (see *e.g.* [2]) and therefore its tree of outcomes is regular. But as pointed in Section III-C topological and probabilistic largeness coincide for ω-regular properties of regular trees and therefore the strategy is topologically good as well. Conversely, any topologically good strategy can be chosen with finite memory thanks to (a small variant of) Theorem 2 (as the winning condition in $\widehat{\mathbb{G}}$ is ω-regular) and therefore the tree of its outcomes is regular. Hence, the same strategy in the probabilistic context is almost surely winning for the same reason as previously. ■

In [7] Varacca and Völzer showed that for finite Markov chains with ω-regular objectives topological and probabilistic largeness coincide. A natural question, addressed by Asarin *et al.* in [8], is whether this is still true for Markov decision processes (*i.e.* a game with Eloise and Nature in the probabilistic setting). For this they introduced a notion of three player

[8] In case $\theta(v)$ is undefined Éloïse looses the play. We assume this never happens but it can easily be captured in the winning condition by adding an extra vertex.

games[9] (EBM-games) where Éloïse plays against Abélard who is split into two sub-players — Banach who is good and Mazur who is evil. Banach starts playing for Abélard and after some time he decides to let Mazur play for a while and then Mazur let him play again and so on. Éloïse *does not observe* who — Banach or Mazur — is acting for Abélard. Say that Éloïse wins the game if she has a strategy such that Banach also has a strategy such that whatever Mazur does the winning condition is satisfied. The main result of [8] is that for an EBM-game on a finite arena with an ω-regular objective Éloïse has a winning strategy iff she has an almost-surely winning strategy in the Éloïse-Nature game obtained by seing the "Banach/Mazur" player as the single stochastic player Nature (for arbitrary probability distributions).

This result is a corollary of Theorem 5 as it is easily seen that in the Éloïse-Nature game obtained by merging the "Banach/Mazur" players as the single player Nature, Éloïse has a topologically good strategy if and only if Éloïse wins the EBM-game[10].

Remark that our approach differs from [8] by the fact that we reason by reduction instead of providing an ad-hoc algorithm; moreover topologically good strategies make sense also for two-player games with Nature while EBM-games do not extend naturally to capture a second antagonistic player.

2) Variant of Tree Automata: A ***parity tree automaton*** $\mathcal{A}$ is a tuple $\langle A, Q, q_{ini}, \Delta, \mathrm{Col}\rangle$ where A is a finite input alphabet, Q is the finite set of states, $q_{\mathrm{ini}} \in Q$ is the initial state, $\Delta \subseteq Q \times A \times Q \times Q$ is the transition relation and $\mathrm{Col} : Q \to C$ is a colouring function.

Given an A-labelled complete binary tree t, a ***run*** of $\mathcal{A}$ over t is a Q-labelled complete binary tree ρ such that (*i*) the root is labelled by the initial state, *i.e.* $\rho(\varepsilon) = q_{\mathrm{ini}}$; (*ii*) for all nodes u, $(\rho(u), t(u), \rho(u\cdot 0), \rho(u\cdot 1)) \in \Delta$. A branch $\pi = b_1 b_2 b_3 \cdots$ is ***accepting*** in the run ρ if it satisfies the parity condition, *i.e.* $\liminf(\mathrm{Col}(\rho(b_1 \cdots b_i)))_{i\geqslant 0}$ is even; otherwise it is rejecting.

Classically, one says that a tree t is accepted by $\mathcal{A}$ if there exists a run of $\mathcal{A}$ on t such that *all* branches in it are accepting. One denotes by $L(\mathcal{A})$ the set of accepted trees and such a language is called ***regular***.

Several relaxations of this criterion have been considered.

- ***Automata with cardinality constraints***. Among others one can consider the language $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$ of those trees for which there is a run with at least uncountably many accepting branches [5], and the language $L^{\mathrm{Rej}}_{\leqslant\mathrm{Count}}(\mathcal{A})$ of those trees for which there is a run with at most countably many rejecting branches [9].
- ***Automata with topological bigness constraints***: a tree belongs to $L^{\mathrm{Acc}}_{\mathrm{Large}}(\mathcal{A})$ if and only if there is a run whose set of accepting branches is large [9].
- ***Qualitative tree automata*** [17]: a tree belongs to $L^{\mathrm{Acc}}_{\mathrm{Large}}(\mathcal{A})$ if and only if there is a run whose set of accepting branches has measure 1.

Our results implies the following theorem [5], [9].

Theorem 6. *For any parity tree automaton $\mathcal{A}$, $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$, $L^{\mathrm{Rej}}_{\leqslant\mathrm{Count}}(\mathcal{A})$ and $L^{\mathrm{Acc}}_{\mathrm{Large}}(\mathcal{A})$ are effectively regular.*

Proof sketch: We only focus on automata with cardinality constraints. Start with the case $L^{\mathrm{Rej}}_{\leqslant\mathrm{Count}}(\mathcal{A})$. One can think of the acceptance of a tree t as a game $\mathbb{G}$ where Éloïse labels the input by transitions and Nature chooses which branch to follow: $t \in L^{\mathrm{Rej}}_{\leqslant\mathrm{Count}}(\mathcal{A})$ iff the leaking value of this game is at most $\aleph_0$. Consider game $\widehat{\mathbb{G}}$ as in Theorem 1. This game (up to some small changes) is essentially the following: the play starts at the root of the tree; in a node u Éloïse chooses a valid transition of the automaton and indicates a direction she wants to avoid and then Abélard chooses the next son; the winning condition is that either the parity condition is satisfied or finitely often Abélard obeys Éloïse. It is then easy to see this latter game as the "usual" acceptance game for some tree automaton with an ω-regular acceptance condition.

Now consider the case $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$. One can think of the acceptance of a tree t as a game $\mathbb{G}$ where Éloïse does nothing, Abélard labels the input by transitions and Nature chooses which branch to follows; the winning condition is the complement of the parity condition: $t \in L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$ iff the leaking value of this game is $2^{\aleph_0}$. Again, one can consider game $\widehat{\mathbb{G}}$ as in Theorem 1 and we know that Abélard has a winning strategy. Then switch the names of the players, complement the winning condition and obtain an acceptance game for $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$ where in a node u Éloïse chooses a valid transition of the automaton, then Abélard indicates a direction he wants to avoid and then Éloïse chooses the next son; the winning condition is that the parity condition is satisfied and infinitely often Éloïse obeys Abélard. Then one can easily prove that this game is equivalent to the following game: in a node u Éloïse chooses a valid transition of the automaton and may indicate a direction to follow, then Abélard chooses the next son (and if Éloïse indicated a direction to follow he must respect it); the winning condition is that the parity condition is satisfied and infinitely often Éloïse does not indicate a direction. This latter game can easily be seen as the "usual" acceptance game for some tree automaton with an ω-regular acceptance condition. ■

3) Games on Infinite Arenas: We claim that, in many contexts where the probabilistic approach fails, the two approaches (cardinality and topological) that we proposed permit to obtain *positive* results for the main problem usually addressed: decide if Éloïse has a "good" strategy and if so compute it. Due to space constraints we only briefly mention some of these contexts and, for each of them, point out the undecidability result in the probabilistic setting and the decidability result in the two-player game (without nature) setting that combined with our main results (Theorem 1 / Theorem 2) leads to decidability in the cardinality/topological setting.

[9] We change here the name of the players to stick to the presentation of this paper and use EBM-game instead of the original name, ABM-game.

[10] She has a topologically good strategy if and only if she has a strategy so that in the induced Banach-Mazur game she has a strategy that wins against any strategy of Abélard: hence, it suffices to see the Éloïse in the Banach-Mazur game as Banach and Abélard as Mazur.

- Games played on pushdown graphs. Undecidable (except under a quite strong restriction) for Éloïse-Nature reachability game in the probabilistic context [3]. Éloïse-Abélard-Nature (resp. Éloïse-Nature) parity games are decidable in the cardinality (resp. topological) setting as a consequence of [18].
- The same holds for much general classes of infinite graphs, *e.g.* the one generated by collapsible pushdown automata [19] (that are meaningful *e.g.* for higher-order program verification).
- A popular non regular winning condition in pushdown game is the boundedness/unboundedness condition that imposes a restriction on how the stack height evolves during a play. For stochastic games with Nature only (*i.e.* probabilistic pushdown automata) there are positive results [20] but they break (because of [3]) whenever Éloïse comes in. In the cardinality (resp. topological) setting we have decidability in the general case of Éloïse-Abélard-Nature (resp. Éloïse-Nature) thanks to Theorem 1 (resp. Theorem 2) combined with the results in [21], [22].

B. Some Consequences in the Imperfect-Information Setting

In the case of finite arena, as soon as one considers co-Büchi conditions almost-sure winning is undecidable even for Éloïse-Nature game where Éloïse is totally blind (all vertices are equivalent) [4]. Thanks to Theorem 3 and 4 and the results in [15] we get decidability for finite arena for any parity condition. A temptation would be to consider cardinality/topological variants of probabilistic automata on infinite words [4] as such a machine can be though as an Éloïse-Nature game where Éloïse is totally blind: *e.g.* declare that an ω-word is accepted by an automaton if all but a countable number of runs on it are accepting (resp. the set of accepting runs is large). However, a simple consequence (omitted here due to space) of our results is that the languages defined in this way are always ω-regular.

There is very few work in the probabilistic setting about games with imperfect information played on *infinite* arenas. The notable exception is the case of concurrent reachability games played on single-exit recursive state machines[11] for which impressive results where obtained in [23]. In the non-stochastic setting, it is easy to derive decidability results for *parity* game played on pushdown graphs when Éloïse perfectly observes the stack content but not the exact control state and Abélard is perfectly informed (see *e.g.* [24]); this result can easily be extended for more general classes of graphs as collapsible pushdown graphs as defined in [19]. Hence, thanks to Theorem 3 and 4 we obtain decidability results for games with Nature played on those classes of infinite arenas. Note that in the cardinality setting, even if we require that Abélard has perfect information our model captures concurrent games.

[11]Concurrency is a special instance of imperfect information where Abélard is perfectly informed: he chooses an action which is stored on the state and cannot be observed by Éloïse who next chooses an action that together with the one by Abélard leads to the next state (chosen by Nature). Recursive state machines are equivalent with pushdown automata; however the single exit case quite strongly restricts the model.

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APPENDIX

Proposition 1. *For any arena, any initial vertex, any pair of strategies $(\varphi_{\rm E}, \varphi_{\rm A})$ and any Borel subset $S \subseteq \text{Outcomes}_{v_0}^{\varphi_{\rm E},\varphi_{\rm A}}$, one has $Card(S) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.*

Proof: As $\text{Outcomes}_{v_0}^{\varphi_{\rm E},\varphi_{\rm A}}$ is the set of branches of a tree whose set of directions is countable (see for instance Theorem 3.11 in [14]), it is a Polish space with the standard basis $\{\text{Cone}_{T_{v_0}^{\varphi_{\rm E},\varphi_{\rm A}}}(v) \mid v \in T_{v_0}^{\varphi_{\rm E},\varphi_{\rm A}}\}$. By [14, Theorem 13.6], any Borel subset S is either countable or has cardinality $2^{\aleph_0}$. ∎

Lemma 1. *Let t be a D-tree for some D and B be a Borel set of branches in t. Then B is large if and only if there exists a dense set of nodes $W \subseteq t$ such that $\mathcal{B}(W) \subseteq B$.*

Proof: Assume that B is large in t and let f be a decomposition-invariant strategy for Éloïse in the associated Banach-Mazur game. Consider the set:

$$W = \{vf(v) \mid v \in t\}.$$

The set W is dense (as for all $v \in t$, $v \sqsubset vf(v) \in W$). We claim that $\mathcal{B}(W)$ is included in B. Let π be a branch in $\mathcal{B}(W)$. As π has infinitely many prefixes in W, there exists a sequence of words $u_1, u_2, \cdots$ such that $u_1 f(u_1) \sqsubset u_2 f(u_2) \sqsubset \cdots \sqsubset \pi$. As the lengths of the u_i are strictly increasing, there exists a sub-sequence $(v_i)_{i\geqslant 1}$ of $(u_i)_{i\geqslant 1}$ such that for all $i \geqslant 1$, $v_i f(v_i) \sqsubset v_{i+1}$. Now, consider the play in the Banach-Mazur game where Abélard first move to v_1 and then Éloïse responds by going to $v_1 f(v_1)$. Then Abélard moves to v_2 (which is possible as $v_1 f(v_1) \sqsubset v_2$) and Éloïse moves to $v_2 f(v_2)$. An so on. In this play Éloïse respects the strategy f and therefore wins. Hence, the branch π associated to this play belongs to B.

Conversely let W be a dense set of nodes such that $\mathcal{B}(W) \subseteq B$. To show that B is large, we define a decomposition-invariant strategy f for Éloïse in the associated Bannach-Mazur game. For all node u we pick v of W such that u is a strict prefix of v (since W is dense there must always exist such a v). Let $v = uu'$ and fix $f(u) = u'$. A play where Éloïse respects f goes through infinitely many nodes in W (as f always points to a vertex in W). Hence, the branch associated with the play belongs to $\mathcal{B}(W) \subseteq B$ which shows that f is winning for Éloïse. ∎

Lemma 2. *A set of nodes W is dense if and only if there exists a direction mapping that points to W.*

Proof: Assume that W is dense. We define $d(u)$ by induction on u as follows. Let u be a node in t such that $d(u)$ is not defined yet: we pick a node $ud_1 \cdots d_k \in W$ (there must exists one since W is dense), and for all $j \leqslant k$ we define

$$d(ud_1 \cdots d_{j-1}) = d_j.$$

The mapping is defined on every node and satisfies the requirement by definition. The other implication is straightforward (for all node u, there exists $ud_1 \cdots d_k \in W$). ∎

Lemma 3. *The set of branches satisfying the parity condition in t is large if and only if there is a valid local-strategy (φ_f, φ_n) such that any (φ_f, φ_n)-compatible branch satisfies the parity condition. Moreover one can choose (φ_f, φ_n) such that $\varphi_f(u_1) = \varphi_f(u_2)$ and $\varphi_n(u_1) = \varphi_n(u_2)$ whenever $t[u_1] = t[u_2]$.*

Proof: As in the proof of Lemma 1 we rely on the characterisation of large sets by means of Banach-Mazur games.

Obviously if there is a valid local-strategy (φ_f, φ_n) such that any (φ_f, φ_n)-compatible branch satisfies the parity condition, then it leads a winning strategy for Éloïse in the Banach-Mazur game. Indeed, for her first move Éloïse goes down in the tree using φ_f until she ends up in a node whose father's second component by φ_f was $\top$ and in the next rounds she does similarly but using φ_n. The resulting play is a (φ_f, φ_n)-compatible branch hence, satisfies the parity condition.

We now prove the other implication, *i.e.* we assume that the set of branches satisfying the parity condition in t is large or equivalently that Éloïse wins the Banach-Mazur game. The beginning of the proof is very similar to the one that Banach-Mazur games with Muller winning condition admit positional strategies [13, Proposition 13]. Let u be a node in t then one denotes by $C(u) = \{\text{Col}(v) \mid u \sqsubseteq v\}$ the set of colours of nodes reachable from u in t. Obviously one has $C(w) \subseteq C(u)$ for all $u \sqsubseteq w$. In case one has $C(w) = C(u)$ for all $u \sqsubseteq w$ we say that u is a *stable* node (and so does its descendants). As the set of colours is finite, for all node u there is a stable node v such that $u \sqsubseteq v$.

We claim that for all stable node u one has $\min C(u)$ even. Indeed, assume that there is some stable u such that $\min C(u) = m$ is odd: then a winning strategy (leading a contradiction) of Abélard in the Banach-Mazur game would consist to go to u in the first move and then whenever he has to play to go to a node with colour m (which he can always do by stability).

Now we define a valid local-strategy (φ_f, φ_n) as follows. First, fix a total ordering on D. For every $u \in t$ call u_s the unique minimal (for prefix ordering) stable node such that $u \sqsubset u_s$: define $\varphi_f(u) = (d, x)$ where $u_s = u \cdot d \cdot w$ with $d \in D$ and $x = \top$ if $w = \varepsilon$ and $x = \bot$ otherwise. For every $u \in t$ that is stable call u' be the unique minimal (for length lexicographic ordering) node with colour $\min C(u)$ and such that $u' \sqsubset u_s$: define $\varphi_n(u) = (d, x)$ where $u_s = u \cdot d \cdot w$ with $d \in D$ and $x = \top$ if $w = \varepsilon$ and $x = \bot$ otherwise. For every $u \in t$ that is not stable define $\varphi_n(u) = (d, \bot)$ where d is the minimal direction such that $ud \in t$ (the value of φ_n does not matter but we want it to be the same in all isomorphic subtrees so we have to define it in a systematic way). From the definition one directly gets that $\varphi_f(u_1) = \varphi_f(u_2)$ and $\varphi_n(u_1) = \varphi_n(u_2)$ whenever $t[u_1] = t[u_2]$.

The fact that (φ_f, φ_n) is valid is by definition and the fact that any (φ_f, φ_n)-compatible branch satisfies the parity condition is a direct consequence of the fact that for all stable node u one has $\min C(u)$ even. ∎

Lemma 3. *The leaking value of $\mathbb{G}$ is at most $\aleph_0$ if and only if Éloïse has a winning strategy in $\widehat{\mathbb{G}}$.*

More precisely, from a winning strategy (resp. positional winning strategy) $\widehat{\varphi}_{\mathrm{E}}$ of Éloïse in $\widehat{\mathbb{G}}$, we can define a strategy (resp. positional winning strategy) φ_{E} for Éloïse in $\mathbb{G}$ such that $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$.

Proof:

First assume that Éloïse has a winning strategy $\widehat{\varphi}_{\mathrm{E}}$ in $\widehat{\mathbb{G}}$. This direction is very similar to the perfect information case. We define a strategy φ_{E} for Éloïse in $\mathbb{G}$ by letting $\varphi_{\mathrm{E}}(\lambda) = \gamma$ whenever $(\gamma, \theta) = \widehat{\varphi}_{\mathrm{E}}(\lambda)$.

Let us now prove that $CardLeak(\varphi_{\mathrm{E}}) \leqslant \aleph_0$. For this, fix a strategy φ_{A} of Abélard in $\mathbb{G}$ and consider a play λ in $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \backslash \Omega$. Let $\hat{\lambda}$ be the corresponding play in $\widehat{\mathbb{G}}$.

As $\widehat{\varphi}_{\mathrm{E}}$ is winning in $\widehat{\mathbb{G}}$, $\hat{\lambda}$ (which respects $\widehat{\varphi}_{\mathrm{E}}$) is won by Éloïse. As $\hat{\lambda}$ does not satisfy the parity condition, Éloïse wins because Abélard obeys Éloïse only finitely often. Let π_λ be the longest prefix π of λ such that $\hat{\pi}$ is of the form $\pi' v(\gamma, \theta) v'$ with $v' \neq \theta(v)$ (*i.e.* it is the last time where Abélard obeys Éloïse).

We claim that $\lambda \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \backslash \Omega$ is uniquely characterised by π_λ. In particular $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \backslash \Omega$ is countable as it can be injectively mapped into the countable set of partial plays in $\mathbb{G}$.

Let $\lambda_1 \neq \lambda_2 \in \mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}},\varphi_{\mathrm{A}}} \backslash \Omega$. We will show that $\pi_{\lambda_1} \neq \pi_{\lambda_2}$. Consider the greatest common prefix π of λ_1 and λ_2. As λ_1 and λ_2 respects the same strategies for Éloïse and Abélard, π must end by some $v\gamma$ with v of Éloïse. In particular there exists $v_1 \neq v_2 \in \Delta_{\mathrm{E}}(v, \gamma)$ such that $\pi v_1 \sqsubset \lambda_1$ and $\pi v_2 \sqsubset \lambda_2$. The partial play $\hat{\pi}$ ends in $v(\gamma, \theta)$ for some $\theta \in \Theta_\gamma$. Assume w.l.o.g. that $\theta(v) \neq v_1$. Abélard obeys Éloïse at $\widehat{\pi v_1}$ in λ_1 and in particular, $\pi v_1 \sqsubseteq \pi_{\lambda_1}$ and therefore $\pi_{\lambda_1} \not\sqsubseteq \pi_{\lambda_2}$. ∎

Theorem 4. *Éloïse has a topologically good strategy in $\mathbb{G}$ if and only if she has a winning strategy in $\widetilde{\mathbb{G}}$.*

More precisely, from a winning strategy (resp. positional winning strategy) $\widetilde{\varphi}_{\mathrm{E}}$ of Éloïse in $\widetilde{\mathbb{G}}$, we can define a topologically good strategy (resp. positional strategy) φ_{E} for Éloïse in $\mathbb{G}$.

Proof: Strategies $\widetilde{\varphi}$ for Éloïse in $\widetilde{\mathbb{G}}$ are in bijections with pairs made of a strategy φ in $\mathbb{G}$ together with a local-strategy (φ_f, φ_n) in the tree of the outcomes of φ in $\mathbb{G}$. Now if $\widetilde{\varphi}$ is winning in $\widetilde{\mathbb{G}}$ we have thanks to the second part of $\widetilde{\Omega}$ that the local-strategy (φ_f, φ_n) is valid, and thanks to the first part of $\widetilde{\Omega}$ that any compatible play is winning for Éloïse in the Banach-Mazur game. Hence, it implies that φ is topologically good (the set of winning plays in $T_{v_0}^{\varphi}$ is large).

Conversely if Éloïse has a topologically good strategy φ in $\mathbb{G}$ we can associate with φ a local-strategy (φ_f, φ_n) as in Lemma 3 (applied to $T_{v_0}^{\varphi}$). Using φ, φ_f and φ_n we define a winning strategy $\widetilde{\varphi}$ for Éloïse in $\widetilde{\mathbb{G}}$ as follows. We let $\widetilde{\varphi}([(v_0, X_0, x_0)(v_1, X_1, x_1) \cdots (v_k, X_k, x_k)]_{/\approx}) = (\gamma, \theta)$ where $\gamma = \varphi([v_0 v_1 \cdots v_k]_{/\sim})$ and θ is defined as follows. Let $v \in V$: if there is no $v'_0 \cdots v'_k \in T_\varphi \cap [v_0 \cdots v_k]_{/\sim}$ with $v_k = v$ we let $\theta(v)$ undefined; otherwise choose such a $v'_0 \cdots v'_k$ (the representative actually does not matter thanks to the fact that (φ_f, φ_n) is the same in isomorphic subtrees) and define $\theta(v) = (\varphi_f(v'_0 \cdots v'_k), \varphi_n(v'_0 \cdots v'_k))$.

Now consider a play $\widetilde{\lambda} = (v_0, X_0, x_0)(v_1, X_1, x_1)(v_2, X_2, x_2) \cdots$ in $\widetilde{\mathbb{G}}$ where Éloïse respects $\widetilde{\varphi}$. Then if there are infinitely many i such that $X_i = E$ then there are infinitely many j such that $X_j = A$ (this is because (φ_f, φ_n) is valid). Moreover if there are infinitely many i such that $X_i = E$ then the play $v_0 v_1 v_2 \cdots$ is a branch in T_φ that is (φ_f, φ_n)-compatible and therefore it satisfies the parity condition by Lemma 3 and definition of (φ_f, φ_n). Therefore, strategy $\widetilde{\varphi}$ is winning for Éloïse in $\widetilde{\mathbb{G}}$ and it concludes the proof. ∎